

# The Second Virial Coefficient of Spin-1/2 Interacting Anyon System

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## Abstract

Evaluating the propagator by the usual time-sliced manner, we use it to compute the second virial coefficient of an anyon gas interacting through the repulsive potential of the form  $g/r^2$  ( $g > 0$ ). All the cusps for the unpolarized spin-1/2 as well as spinless cases disappear in the  $\omega \rightarrow 0$  limit, where  $\omega$  is a frequency of harmonic oscillator which is introduced as a regularization method. As  $g$  approaches to zero, the result reduces to the noninteracting hard-core limit.

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Since the anyon whose statistics interpolates between boson and fermion at two-dimensions [1–3] was introduced, until recently the main focus has been on the free anyon gas, i.e., noninteraction apart from statistical interaction of Aharonov-Bohm type. In order to investigate the statistical properties of a free anyon gas the thermodynamic quantities such as the second virial coefficient as a function of statistical parameter  $\alpha$  has been calculated for both spinless [4,5] and spin-1/2 cases [6]. The second virial coefficient of the spinless case shows the periodic dependence on  $\alpha$  and nonanalytic behavior at bose points. However, for spin-1/2 case the discontinuities appear at bose points and periodicity is also removed. This difference comes from the fact that the introduction of spin allows the irregular wavefunction at origin. Even if no irregular solution is assumed in the spin-1/2 case, the cusps exist at all integer points. Recently we calculated the second virial coefficient for spinless and spin-1/2 free anyon gases [7] for the various values of self-adjoint extension [8] parameter. The result for spin-1/2 case exhibits a completely different cusp and discontinuity structure from Ref. [6], due to the different condition for the occurrence of the irregular wavefunction at origin.

Loss and Fu [9] studied the interacting anyon gas with a repulsive potential of the form  $g/r^2$  ( $g > 0$ ), using the similar regularization procedure to that used in Ref. [4]. They chose  $1/r^2$ -potential, because it does not remove the scale invariance of theory and the path-integral solution is simply obtained. Furthermore, the probability of the overlap of two particles is always zero. This property is also valid for spin-1/2 system with the same two-particle interaction [10]. They showed that this simple interaction makes the cusps at bose points smooth for spinless case.

In this paper, we will compute the second virial coefficient of the spin-1/2 anyon gas interacting through this repulsive potential using the harmonic oscillator regularization. It is found that the cusps at both boson and fermion points of the second virial coefficient calculated under the condition that no irregular solution is assumed become smooth as in the case of Loss and Fu [9]. As  $g \rightarrow 0$ , the nonanalytic behavior of Blum *et al.* [6] is reproduced.

We begin with the kernel for the anyon system with  $g/r^2$  and harmonic oscillator interactions

$$K[\mathbf{r}_f, \mathbf{r}_i; T] = \int D\mathbf{r} e^{i \int_0^T dt L(\mathbf{r}, \dot{\mathbf{r}}, t)}, \quad (1)$$

where

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{M}{2} \dot{\mathbf{r}}^2 - \alpha \dot{\theta} - \frac{g}{r^2} - \frac{M}{2} \omega^2 \mathbf{r}^2 \quad (2)$$

is the Lagrangian of the system. Following the similar procedure to Ref. [11], one can obtain the euclidean kernel as

$$\begin{aligned} G[\mathbf{r}_f, \mathbf{r}_i; \tau] &= \sum_{m=-\infty}^{\infty} e^{im(\theta_f - \theta_i)} G_m[r_f, r_i; \tau], \\ G_m[r_f, r_i; \tau] &= \frac{M\omega}{2\pi \sinh \omega\tau} \exp \left[ -\frac{M\omega}{2} \frac{\cosh \omega\tau}{\sinh \omega\tau} (r_i^2 + r_f^2) \right] \\ &\quad \times I_{\sqrt{(m+\alpha)^2 + 2gM}} \left( \frac{M\omega r_i r_f}{\sinh \omega\tau} \right) \end{aligned} \quad (3)$$

where  $I_\nu(x)$  is the modified Bessel function and  $\tau = iT$ . Then we perform the Laplace transform to obtain the energy-dependent Green's function:

$$\begin{aligned} \hat{G}[\mathbf{r}_f, \mathbf{r}_i; E] &= \sum_{m=-\infty}^{\infty} e^{im(\theta_f - \theta_i)} \hat{G}_m[r_f, r_i; E], \\ \hat{G}_m[r_f, r_i; E] &= \frac{1}{2\pi\omega r_i r_f} \frac{\Gamma \left( [1 + \sqrt{(m+\alpha)^2 + 2gM} + E/\omega]/2 \right)}{\Gamma \left( 1 + \sqrt{(m+\alpha)^2 + 2gM} \right)} \\ &\quad \times W_{-\frac{E}{2\omega}, \sqrt{(m+\alpha)^2 + 2gM}} \left( M\omega [\text{Max}(r_i, r_f)]^2 \right) \\ &\quad \times M_{-\frac{E}{2\omega}, \sqrt{(m+\alpha)^2 + 2gM}} \left( M\omega [\text{Min}(r_i, r_f)]^2 \right) \end{aligned} \quad (4)$$

where  $W_{\kappa, \mu}(x)$  and  $M_{\kappa, \mu}(x)$  are the usual Whittaker's functions, and  $\text{Max}(x, y)$  and  $\text{Min}(x, y)$  are the maximum and minimum values of  $x$  and  $y$ , respectively. From the poles of the Green's function, the bound state spectrum of the system is straightforwardly obtained:

$$E_{n,m} = \left( 2n + 1 + \sqrt{(m+\alpha)^2 + 2gM} \right) \omega. \quad (5)$$

The plot of  $E_{0,0}$  at  $gM = 1$  is shown in figure 1. The cusps that happened in the absence of  $1/r^2$  potential disappear and become smooth. Therefore, by the introduction of repulsive potential, we expect that the nonanalytic dependence on  $\alpha$  in various thermodynamic quantities would be suppressed.

Now, we calculate the second virial coefficient  $B_2$  of this system. The two-particle partition function  $Z_2$  is given by

$$\begin{aligned} Z_2 &\equiv \text{Tr} \exp(-\beta H_2) \\ &= 2A\lambda_T^2 Z_{rel}, \end{aligned} \quad (6)$$

where  $H_2$  is the two-particle Hamiltonian,  $\beta = 1/kT$ ,  $A$  is the area of the system,  $\lambda_T = (2\pi/kTM)^{1/2}$  is the thermal de Broglie wavelength, and  $Z_{rel}$  is the partition function in relative coordinates. The second virial coefficient then is

$$\begin{aligned} B_2(\alpha, T) &= \frac{A}{2} - 2\lambda_T^2 Z_{rel} \\ &= \frac{A}{2} - 2\lambda_T^2 \sum_{n,m} e^{-\beta E_{n,m}}, \end{aligned} \quad (7)$$

where  $E_{n,m}$  is given in Eq. (5) and  $M$  is replaced by  $2M$ . The summation over even (odd)  $m$ 's corresponds to the boson (fermion) statistics. At first performing the summation over  $n$ , we obtain

$$B_2(\alpha, T) = \frac{A}{2} - \frac{\lambda_T^2}{\sinh \beta \omega} \sum_m e^{-\beta \sqrt{(m+\alpha)^2 + gM\omega}}. \quad (8)$$

Consider the spinless case by summing only over even  $m$ 's. In order to regularize the infinite area, we calculate  $B_2(\alpha, T) - B_2(\alpha = 0, T)$  :

$$B_2(\alpha, T) - B_2(0, T) = \frac{\lambda_T^2}{\sinh \beta \omega} \sum_{m=\text{even}} \left[ e^{-\beta \sqrt{m^2 + gM\omega}} - e^{-\beta \sqrt{(m+\alpha)^2 + gM\omega}} \right]. \quad (9)$$

The result in the  $\omega \rightarrow 0$  limit is just that of Ref. [9].

Next, consider the unpolarized spin-1/2 anyon case. This can be done by averaging over four possible spin states:

$$B_2(\alpha, T) - \bar{B}_2(0, T) = \frac{\lambda_T^2}{4 \sinh \beta \omega} \left\{ 3 \sum_{m=odd} \left[ e^{-\beta \sqrt{m^2 + gM\omega}} - e^{-\beta \sqrt{(m+\alpha)^2 + gM\omega}} \right] + \sum_{m=even} \left[ e^{-\beta \sqrt{m^2 + gM\omega}} - e^{-\beta \sqrt{(m+\alpha)^2 + gM\omega}} \right] \right\}, \quad (10)$$

where  $\bar{B}_2(0, T)$  is the averaged  $B_2(0, T)$  which cannot be determined but has no  $\alpha$ -dependence. We show  $\omega \rightarrow 0$  limit of  $B_2(\alpha, T) - \bar{B}_2(0, T)$  as a function of  $\alpha$  for  $g = 0, 0.05, 0.1$  and 1 in figure 2. When  $g > 0$ , the second virial coefficient has no cusps for all  $\alpha$  as expected. As  $g \rightarrow 0$ , the previous result [6] is reproduced:  $|\alpha| - 2\alpha^2$  for boson point and  $3|\alpha| - 2\alpha^2$  for fermion point [12]. As a result, the repulsive interaction removes all the cusps for spin-1/2 case as well as the spinless one.

Even though  $1/r^2$ -potential is adopted to study the interacting anyons due to its simplicity, more realistic interaction between anyons should be introduced in order to apply to real physical systems. If we think anyons as the particles both carrying magnetic flux and electrical charge, the consideration of Coulomb interaction arises naturally. We have already calculated the kernel and bound states for Aharonov-Bohm-Coulomb system incorporating the self-adjoint extension method into the Green's function formalism appropriately [13]. Though the simple harmonic oscillator regularization seems to be impossible because of difficulty in getting the path-integral solution for Aharonov-Bohm-Coulomb plus harmonic oscillator system, the second virial coefficient may be obtained from the appropriate phase shift method in scattering theory. This problem is now under study.

In conclusion, we find the path-integral kernel for the interacting spin-1/2 anyons with repulsive potential and harmonic oscillator, and calculate the second virial coefficient using the partition function obtained by summing the harmonic oscillator bound states. For unpolarized spin-1/2 anyons, all the cusps at both boson and fermion points disappear, as in the spinless case. The nonanalytic behavior with  $\alpha$  is reproduced when  $g \rightarrow 0$ .

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## FIGURES

FIG. 1. The bound state energy as a function of  $\alpha$  when  $n, m = 0$ . Solid line :  $gM = 1$  case. Dashed line :  $g = 0$  case in the presence of irregular solution. Dotted line :  $g = 0$  case in the absence of irregular solution.

FIG. 2.  $[B_2(\alpha, T) - \bar{B}_2(0, T)]/\lambda_T^2$  as a function of  $\alpha$  at various  $gM$  values. Thick solid line :  $gM = 0$ . Dotted line :  $gM = 0.05$ . Short-dotted line :  $gM = 0.1$ . Thin solid line :  $gM = 1$ .





